A simple method to locate the optimal solution for exponentially deteriorating items under trade credit financing

Kuo-Nan Huang\textsuperscript{a,*}, Jui-Jung Liao\textsuperscript{b}

\textsuperscript{a}Department of International Trade, St. John’s University, Tamsui, Taipei Country 251, Taiwan, ROC
\textsuperscript{b}Department of Business Administration, Chihlee Institute of Technology, Taipei, Taiwan, ROC

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Abstract

In this paper, we discuss a paper of Chang and Teng [C.T. Chang, J.T. Teng, Retailer’s optimal ordering policy under supplier credits, Mathematical Methods of Operations Research 60 (2004) 471–483], they established an inventory model for deteriorating items when the supplier permits not only a cash discount but also a permissible delay. They also proved a closed-form solution of the inventory system under their assumption. Herein, we consider a deteriorating item that follows an exponential distribution. Thus, Chang and Teng [C.T. Chang, J.T. Teng, Retailer’s optimal ordering policy under supplier credits, Mathematical Methods of Operations Research 60 (2004) 471–483] is a special case in our model. The main purpose of this paper is threefold: First, we show that the total variable cost per unit time is convex by a rigorous proof. Second, with convexity, the optimal solution procedures to find the optimal ordering policy, which is independent of Chang and Teng’s assumption, and bounds for the optimal ordering time are provided. Third, we compare optimal solutions obtained by using our approach and Chang and Teng’s approach. Finally, sensitivity analysis is performed to study the effects of changing parameters values on the optimal solution.

Keywords: Inventory; Cash discount; Delay payments; Deteriorating items; Trade credit

1. Introduction

The practice of suppliers offering special incentives to retailers for a limited time period to increase demand or decrease inventory is quite prevalent in some industries. In doing this, the supplier-to-retailer incentives can take on many forms, with a cash discount which can encourage the retailer to pay cash on delivery, reduce the default risk and a delay of payment being the most prevalent. Thereafter, many research articles appeared which deal with an extended EOQ model under cash discount or payment delay like Aggarwal and Jaggi [2], Arcelus and Srinivasan [3], Chang [4], Chung and Liao [5,6], Jaggi and Aggarwal [7], Jamal et al. [8,9], Salameh et al. [10], Liao [11,12], Sarker et al. [13,14], Shah [15], Shinn [16], Teng [17] and their references. However, the above literature may explain the total cost (profit) function of the inventory model is convex (concave) or not. In fact, if the total annual cost (profit)
Notation

\(D\)  
the demand rate per year.

\(h\)  
the unit holding cost per year excluding interest charges.

\(p\)  
the selling price per unit.

\(c\)  
the unit purchasing cost, with \(c < p\).

\(I_c\)  
the interest charged per $ in stocks per year by the supplier or a bank.

\(I_d\)  
the interest earned per $ per year.

\(S\)  
the ordering cost per order.

\(Q\)  
the order quantity.

\(r\)  
the cash discount rate, \(0 < r < 1\).

\(\theta\)  
the inventory deterioration rate (constant rate of deterioration).

\(M_1\)  
the period of cash discount.

\(M_2\)  
the period of permissible delay in settling account, with \(M_2 > M_1\).

\(T\)  
the ordering time interval.

\(I(t)\)  
the level of inventory at time, \(0 \leq t \leq T\).

\(Z(T)\)  
the total relevant cost per year.

function of the inventory model is convex (concave), it is easier to find the optimal solution by using the convexity (concavity) property. In the following are some examples:

1. Aggarwal and Jaggi [18] examined the ordering policy of deteriorating items under a permissible delay in payments. Chu et al. [19] showed the total cost function is piecewise-convex but not convex in general and present a simple solution procedure.

2. Hwang and Shinn [20] considered the retailer’ pricing and lot sizing policy for exponentially deteriorating products under the condition of a permissible delay in payments. Chung et al. [21] showed that the annual net profit function is concave and present a simple solution procedure.

3. Goyal [22] investigated an economic item order quantity model under conditions of a permissible delay in payments. Chung [23] showed that the total annual cost function is convex and present a simple solution procedure.

Following this trend, we consider in this paper the same mathematical model of the inventory ordering problem for deteriorating items when the supplier provides not only a cash discount but also a permissible delay as Chang and Teng [1]. They concentrated on the assumption \(\theta T\) is sufficiently small and used a truncated Taylor series expansion for the exponential term in the total cost function per unit time. This approach is commonly used in most research projects in order to derive a closed-form solution. However, their proof about the convexity of the variable cost per unit time is not perfect. In addition, in a real situation, for some deteriorating items such as steel, hardware, glassware, and toys, the rate of deterioration is low and some items such as blood, fish, strawberry, alcohol, gasoline, radioactive chemical and medicine deteriorate rapidly. Thus, this model was extended to consider the exponential distribution deterioration by Ghare and Schrader [24]. Hence, Chang and Teng [1] is a special case in our model.

Based on the above arguments, the purpose of this paper is threefold: First, to show that the variable cost per unit time is convex by a rigorous proof. Second, with convexity, to explore an alternative approach on the optimal solution procedure of the inventory model and to derive the bounds for the optimal replenishment time, thereby an efficient algorithm to locate the optimal replenishment time will be developed. Third, we compare optimal solutions obtained by using our approach and Chang and Teng’s approach. Finally, sensitivity analysis is performed to study the effect of changing parameters values on the optimal solution of the inventory model.

2. The model

For easy tractability, the same assumptions and notation of Chang and Teng [1] are adopted here.

Assumption

(1) The demand for the item is constant with time.

(2) Shortages are not allowed.

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Replenishment is instantaneous.

The distribution of time to the deterioration of the items follows an exponential distribution with parameter \( \theta \), where \( \theta \geq 0 \).

(This assumption is different from Chang and Teng [1].)

During the time the account is not settled, generated sales revenue is deposited in an interest bearing account. At the end of this period (i.e. \( M_1 \) or \( M_2 \)), the customer pays the supplier the total amount in the interest bearing account, and then starts paying off the amount owed to the supplier whenever the customer has money obtained from sales.

Time horizon is infinite.

Moreover, the total relevant cost consists of (a) cost of placing orders, (b) cost of purchasing units, (c) cost of carrying inventory (excluding interest charges), (d) cash discount earned if the payment is made at \( M_1 \), (e) interest earned from sales revenue during the permissible period \([0, M_i]\) (i.e. \( M_1 \) or \( M_2 \)) and (f) cost of interest charges for unsold items after the permissible delay \([0, M_i]\) (i.e. \( M_1 \) or \( M_2 \)) which the cost of interest charges for unsold items after the permissible delay \([0, M_i]\) (i.e. \( M_1 \) or \( M_2 \)) is considered that the customer buys \( I(0) \) units at time 0, and owns \( c(1 - r)I(0) \) or \( cI(0) \) to the supplier. At time \([0, M_i]\) (i.e. \( M_1 \) or \( M_2 \)), the customer sells \( DM_i \) (i.e. \( M_1 \) or \( M_2 \)) units in total, and has \( pDM_i \) (i.e. \( M_1 \) or \( M_2 \)) plus interest earned \( pI_dDM_i^2/2 \) (i.e. \( M_1 \) or \( M_2 \)) to pay the supplier. Thereafter, the customer need to finance \( c(1 - r)I(0) - (pDM_i + pI_dDM_i^2/2) \) (at interest rate \( I_c \)) at time \( M_1 \) or \( M_2 \), and pay the supplier in full in order to get the cash discount. Chang and Teng found that the total variable costs are given by:

**Case 1: \( T \geq M_1 \)**

\[
Z_1(T) = \frac{S}{T} + \frac{D[h + c\theta(1 - r)]}{\theta^2T}(e^{\theta T} - 1) - \frac{hD}{\theta} - \frac{pI_dD}{2T}M_1^2 \\
+ \frac{I_c}{2pDT} \left[ \frac{(c(1 - r)D}{\theta}(e^{\theta T} - 1) - pDM_1(1 + I_dM_1/2) \right]^2.
\]

**Case 2: \( T < M_1 \)**

\[
Z_2(T) = \frac{S}{T} + \frac{D[h + c\theta(1 - r)]}{\theta^2T}(e^{\theta T} - 1) - \frac{hD}{\theta} - pI_dD \left[ M_1 - \frac{T}{2} \right].
\]

**Case 3: \( T \geq M_2 \)**

\[
Z_3(T) = \frac{S}{T} + \frac{D(h + c\theta)}{\theta^2T}(e^{\theta T} - 1) - \frac{hD}{\theta} + \frac{I_c}{2pDT} \left[ \frac{cD}{\theta}(e^{\theta T} - 1) - pDM_2(1 + I_dM_2/2) \right]^2 \\
- \frac{pI_dD}{2T}M_2^2.
\]

**Case 4: \( T < M_2 \)**

\[
Z_4(T) = \frac{S}{T} + \frac{D(h + c\theta)}{\theta^2T}(e^{\theta T} - 1) - \frac{hD}{\theta} - pI_dD \left[ M_2 - \frac{T}{2} \right].
\]

Therefore, the total relevant cost is given by

\[
Z(T) = \begin{cases} 
Z_1(T) & \text{if } T \geq M_1 \\
Z_2(T) & \text{if } 0 < T \leq M_1
\end{cases}
\]

or

\[
Z(T) = \begin{cases} 
Z_3(T) & \text{if } T > M_2 \\
Z_4(T) & \text{if } 0 < T \leq M_2
\end{cases}
\]

At \( T = M_1 \), we find that \( Z_1(M_1) > Z_2(M_1) \). Likewise, at \( T = M_2 \), we have \( Z_3(M_2) > Z_4(M_2) \) as well. Consequently, \( Z(T) \) is continuous except \( T = M_1 \) and \( M_2 \), respectively.

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In reality, Chang and Teng established a closed-form solution based on the assumption that \(\theta T\) is sufficiently small. For doing this, they neglected the third or higher order terms in the expansion of \(e^{\theta T}\), and they did not give the proof of the convexity of \(Z_i(T)\) \((i = 1, 2, 3\) and \(4)\). Herein, we will show the total relevant cost per unit time is convex without any assumptions. For the completeness of the discussion, the total relevant cost per unit time for each case can be represented as the following:

\[
Z_i(T) = \frac{S}{T} + \frac{Dh + B_i\theta^2}{\theta^2 T} (e^{\theta T} - 1) - \frac{hD}{\theta} - \frac{pI_d D}{2T} \bar{M}_i^2 + \frac{I_c}{2p DT} [B_i (e^{\theta T} - 1) - A_i]^2
\]  

(7)

for \(i = 1, 3\), where \(A_i = p D \bar{M}_i (1 + I_d / \bar{M}_i) / 2\), \(B_i = c(1 - \bar{r}_i) D / \bar{M}_i\), \(\bar{M}_1 = M_1\), \(\bar{M}_3 = M_2\), \(\bar{r}_1 = r\) and \(\bar{r}_3 = 0\) and

\[
Z_j(T) = \frac{S}{T} + \frac{Dh + B_j\theta^2}{\theta^2 T} (e^{\theta T} - 1) - \frac{hD}{\theta} - pI_d D \left( \bar{M}_j - \frac{T}{2} \right)
\]  

(8)

for \(i = 2, 4\), where \(B_j = c(1 - \bar{r}_j) D / \bar{M}_j\), \(\bar{M}_2 = M_1\), \(\bar{M}_4 = M_2\), \(\bar{r}_2 = r\) and \(\bar{r}_4 = 0\).

In addition, for Case \(i\) \((i = 1, 3)\), they consider the case: \(p D \bar{M}_i + p I_d D \bar{M}_i^2 / 2 < c(1 - \bar{r}_i) I(0)\) which implies that \(T > \ln(W_i / B_i) / \theta\) where \(W_i = A_i + B_i\) for \(i = 1, 3\).

We first consider Cases 1 and 3 and treat both cases to be defined on \(T > 0\). Then we have

\[
Z_i'(T) = -\frac{S}{T^2} + \frac{Dh + B_i\theta^2}{\theta^2 T^3} (\theta T e^{\theta T} - e^{\theta T} + 1) + \frac{pI_d D \bar{M}_i^2}{2T^2} + \frac{I_c}{2p DT^2} [2B_i \theta T e^{2\theta T} - B_i^2 e^{2\theta T}]
\]  

(9)

\[= -2B_i W_i \theta T e^{\theta T} + 2B_i W_i e^{\theta T} - W_i^2 \]  

and

\[
Z_i''(T) = \frac{2S}{T^3} + \frac{2(Dh + B_i\theta^2)}{\theta^2 T^3} \left( e^{\theta T} - 1 - \theta T e^{\theta T} + \frac{1}{2} \theta^2 T^2 e^{\theta T} \right) - \frac{pI_d D \bar{M}_i^2}{T^3} + \frac{I_c}{2p DT^3} \times \left[ 4B_i^2 (\theta T)^2 e^{2\theta T} - 4B_i^2 (\theta T)^2 e^{2\theta T} + 2B_i^2 e^{2\theta T} - 2B_i W_i (\theta T)^2 e^{\theta T} + 4B_i W_i (\theta T) e^{\theta T} 
\]  

\[= -4B_i W_i e^{\theta T} + 2W_i^2 \]  

(10)

Before proving that both \(Z_1(T)\) and \(Z_3(T)\) are convex on their appropriate domains, we need the following lemmas.

**Lemma 1.** \(e^{\theta T} - 1 - \theta T e^{\theta T} + (\theta^2 T^2 e^{\theta T} / 2) > 0\) for all \(T > 0\).

**Proof.** Let \(g(x) = e^x - 1 - xe^x + (x^2 e^x / 2)\) if \(x > 0\), then \(g'(x) = e^x - x e^x + x^2 e^x / 2 > 0\). Hence \(g(x)\) is increasing for all \(x > 0\). Consequently \(g(x) > g(0) = 0\) if \(x > 0\). We have \(g(x) = e^x - 1 - xe^x + x^2 e^x / 2 > 0\). Let \(x = \theta T\). Then \(e^{\theta T} - 1 - \theta T e^{\theta T} + (\theta^2 T^2 e^{\theta T} / 2) > 0\) if \(T > 0\). This completes proof. \(\Box\)

**Lemma 2.** If \(3B_i > A_i\) for \(i = 1, 3\), then

\[4B_i^2 (\theta T)^2 e^{2\theta T} - 4B_i^2 (\theta T)^2 e^{2\theta T} + 2B_i^2 e^{2\theta T} - 2B_i W_i (\theta T)^2 e^{\theta T} + 4B_i W_i (\theta T) e^{\theta T} - 4B_i W_i e^{\theta T} + 2W_i^2 > 0\]

for all \(T > 0\).

**Proof.** Let \(f(x) = 4B_i^2 x^2 e^{2x} - 4B_i^2 x^2 e^{2x} + 2B_i^2 e^{2x} - 2B_i W_i x^2 e^{x} + 4B_i W_i x e^{x} - 4B_i W_i e^{x} + 2W_i^2 - 2(p D \bar{M}_i)^2,\) for \(i = 1, 3\), then \(f'(x) = 2B_i x^2 e^{4B_i e^{x} - W_i}\). Further, let \(h(x) = 4B_i e^{x} - W_i\), then \(h'(x) = 4B_i e^{x} > 0\). Hence \(h(x)\) is increasing for all \(x > 0\). Moreover, if \(3B_i > A_i\), \(h(x) > h(0) > 0\) for all \(x > 0\) which implies that if \(3B_i > A_i\), \(f'(x) > 0\) for all \(x > 0\). Consequently, if \(3B_i > A_i\), \(f(x) > f(0) > 0\) for all \(x > 0\). Let \(x = \theta T\), then if \(3B_i > A_i\), \(4B_i^2 (\theta T)^2 e^{2\theta T} - 4B_i^2 (\theta T)^2 e^{2\theta T} + 2B_i^2 e^{2\theta T} - 2B_i W_i (\theta T)^2 e^{\theta T} + 4B_i W_i (\theta T) e^{\theta T} - 4B_i W_i e^{\theta T} + 2W_i^2 - 2(p D \bar{M}_i)^2 > 0\) for \(T > 0\). This completes the proof. \(\Box\)
Based upon the above arguments, we have

\[
\frac{d^2 Z_i(T)}{dT^2} > \frac{2S}{T^3} + \frac{2(Dh + B_i \theta^2)}{\theta^2 T^3} \left( e^{\theta T} - 1 - \theta T e^{\theta T} + \frac{1}{2} \theta^2 T^2 e^{\theta T} \right) \\
+ \frac{I_c}{2pDT^3} [4B_i^2 (\theta T)^2 e^{2\theta T} - 4B_i^2 (\theta T^2) e^{2\theta T} \\
+ 2B_i^2 e^{2\theta T} - 2B_i W_i (\theta T)^2 e^{\theta T} + 4B_i W_i (\theta T) e^{\theta T} - 4B_i W_i e^{\theta T} + 2W_i^2 - 2(pDM_i)^2] \\
> 0
\]

for \( i = 1, 3 \). **Lemmas 1 and 2** imply that if \( 3B_i > A_i \), then \( d^2 Z_i(T)/dT^2 > 0 \) if \( T > 0 \) (\( i = 1, 3 \)). Consequently, if \( 3B_i > A_i \), \( Z_i(T) \) (\( i = 1, 3 \)) is convex on \( (0, \infty) \), respectively.

**Lemma 3.** If \( 3B_i \leq A_i \) for \( i = 1, 3 \), we cannot consider the minimum problem for the total variable cost per unit time.

Combining **Lemmas 2** and **3**, in the rest of our mathematical analysis, we assume that \( 3B_i > A_i \) for \( i = 1, 3 \). On the other hand, we have

\[
Z_j'(T) = -\frac{S}{T^2} + \frac{Dh + B_j \theta^2}{\theta^2 T^2} (\theta T e^{\theta T} - e^{\theta T} + 1) + \frac{pI_d D}{2} \quad j = 2, 4
\]

and

\[
Z_j''(T) = \frac{2S}{T^3} + \frac{2(Dh + B_j \theta^2)}{\theta^2 T^3} \left( e^{\theta T} - 1 - \theta T e^{\theta T} + \frac{1}{2} \theta^2 T^2 e^{\theta T} \right). 
\]

Likewise, **Lemma 1** implies that \( Z_j''(T) > 0 \) (\( j = 2, 4 \)) if \( T > 0 \). Consequently, \( Z_j(T) \) (\( j = 2, 4 \)) is convex on \( (0, \infty) \) as well.

### 3. The optimization procedure

In this section, we will present an alternative solution procedure to locate \( T^* \) no matter whether \( \theta T \) is large or not. Hence, consider the following equations:

\[
Z_i'(T) = 0 \quad \text{for} \quad i = 1, 3
\]

and

\[
Z_j'(T) = 0 \quad \text{for} \quad j = 2, 4.
\]

If the root of Eq. (13) or (14) exists, then it is unique. Let \( T_i^* \) (\( i = 1, 3 \)) be the root of Eq. (13) and let \( T_j^* \) (\( j = 2, 4 \)) be the root of Eq. (14). Since \( \lim_{T \to 0^+} Z_j'(T) = -\infty \) (\( j = 2, 4 \)), if \( Z_j'(\tilde{M}_j) > 0 \) (\( j = 2, 4 \)), then \( T_j^* \) (\( j = 2, 4 \)) exists and \( 0 < T_j^* < \tilde{M}_j \) (\( j = 2, 4 \)). On the other hand, if \( Z_j'(\hat{M}_i) < 0 \) (\( i = 1, 3 \)), then \( T_i^* \) (\( i = 1, 3 \)) exists and \( \hat{M}_i < T_i^* \) (\( i = 1, 3 \)). Let \( T^* \) denote the minimum point of \( Z(T) \).

and \( \bar{W}_i = \ln(W_i/B_i)/\theta \) for \( i = 1, 3 \)

\[
\Delta_i = \frac{S}{M_i^2} + \frac{Dh + B_i \theta^2}{\theta^2 M_i^2} (\theta \bar{M}_i e^{\theta \bar{M}_i} - e^{\theta \bar{M}_i} + 1) + \frac{pI_d D}{2} + \frac{I_c}{2pDM_i^2} [2B_i^2 \theta \bar{M}_i e^{2\theta \bar{M}_i} \\
- B_i^2 e^{2\theta \bar{M}_i} - 2B_i W_i \theta \bar{M}_i e^{\theta \bar{M}_i} + 2B_i W_i e^{\theta \bar{M}_i} - W_i^2] \quad i = 1, 3
\]

\[
\Delta_j = \frac{S}{W_j^2} + \frac{Dh + B_j \theta^2}{\theta^2 W_j^2} (\theta \bar{W}_j e^{\theta \bar{W}_j} - e^{\theta \bar{W}_j} + 1) + \frac{pI_d D\bar{M}_j^2}{2W_j^2} + \frac{I_c}{2pDW_j^2} [2B_j^2 \theta \bar{W}_j e^{2\theta \bar{W}_j} \\
- B_j^2 e^{2\theta \bar{W}_j} - 2B_j W_i \theta \bar{W}_j e^{\theta \bar{W}_j} + 2B_j W_i e^{\theta \bar{W}_j} - W_j^2] \quad i = 1, 3
\]
and
\[
\Delta_j = -\frac{S}{M_j^2} + \frac{Dh + B_j\theta^2}{\theta^2 M_j^2} (e^{\theta M_j} - e^{\theta M_i} + 1) + \frac{pl_d D}{2} \quad j = 2, 4.
\] (17)

Eqs. (15)–(17) yield
\[
\Delta_i < 0 \quad \text{if and only if} \quad Z_i'(M) < 0 \quad \text{if and only if} \quad T_i^* > M \quad (i = 1, 3). \tag{18}
\]
\[
\Delta_i^* < 0 \quad \text{if and only if} \quad Z_i'(\tilde{W}_i) < 0 \quad \text{if and only if} \quad T_i^* > \tilde{W}_i \quad (i = 1, 3). \tag{19}
\]
\[
\Delta_j < 0 \quad \text{if and only if} \quad Z_j'(M) < 0 \quad \text{if and only if} \quad T_j^* > M \quad (j = 2, 4). \tag{20}
\]

Then we have the following results.

**Theorem 1.** Suppose that the payment is paid at time $M_1$ and $\tilde{W}_1 > M_1$.

(A) If $\Delta_1 \leq \Delta_2$, then

(A1) If $\Delta_1 < 0$, $\Delta_1^* < 0$ and $\Delta_2 < 0$, then $Z(T^*) = \min\{Z(M_1), Z(T_1^*)\}$ and $T^* = M_1$ or $T_1^*$ associated with the least cost.

(A2) If $\Delta_1 < 0$, $\Delta_1^* \geq 0$ and $\Delta_2 < 0$, then $Z(T^*) = \min\{Z(M_1), Z(\tilde{W}_1)\}$ and $T^* = M_1$ or $\tilde{W}_1$ associated with the least cost.

(A3) If $\Delta_1 < 0$, $\Delta_1^* < 0$ and $\Delta_2 \geq 0$, then $Z(T^*) = \min\{Z(T_2^*), Z(T_1^*)\}$ and $T^* = T_2^*$ or $T_1^*$ associated with the least cost.

(A4) If $\Delta_1 < 0$, $\Delta_1^* \geq 0$ and $\Delta_2 \geq 0$, then $Z(T^*) = \min\{Z(T_2^*), Z(\tilde{W}_1)\}$ and $T^* = T_2^*$ or $\tilde{W}_1$ associated with the least cost.

(A5) If $\Delta_1 \geq 0$, $\Delta_1^* \geq 0$ and $\Delta_2 \geq 0$, then $Z(T^*) = Z(T_2^*)$ and $T^* = T_2^*$.

(B) If $\Delta_1 > \Delta_2$, then

(B1) If $\Delta_1 < 0$, $\Delta_1^* < 0$ and $\Delta_2 < 0$, then $Z(T^*) = \min\{Z(M_1), Z(T_1^*)\}$ and $T^* = M_1$ or $T_1^*$ associated with the least cost.

(B2) If $\Delta_1 < 0$, $\Delta_1^* \geq 0$ and $\Delta_2 < 0$, then $Z(T^*) = \min\{Z(M_1), Z(\tilde{W}_1)\}$ and $T^* = M_1$ or $\tilde{W}_1$ associated with the least cost.

(B3) If $\Delta_1 \geq 0$, $\Delta_1^* \geq 0$ and $\Delta_2 \geq 0$, then $Z(T^*) = Z(T_2^*)$ and $T^* = T_2^*$.

(B4) If $\Delta_1 \geq 0$, $\Delta_1^* \geq 0$ and $\Delta_2 < 0$, then $Z(T^*) = Z(M_1)$ and $T^* = M_1$.

**Proof.** See Appendix A.

**Theorem 2.** Suppose that the payment is paid at time $M_1$ and $\tilde{W}_1 \leq M_1$.

(A) If $\Delta_1 \leq \Delta_2$, then

(A1) If $\Delta_1 < 0$ and $\Delta_2 < 0$, then $Z(T^*) = \min\{Z(M_1), Z(T_1^*)\}$ and $T^* = T_1^*$ or $M_1$ associated with the least cost.

(A2) If $\Delta_1 < 0$ and $\Delta_2 \geq 0$, then $Z(T^*) = \min\{Z(T_2^*), Z(T_1^*)\}$ and $T^* = T_2^*$ or $T_1^*$ associated with the least cost.

(A3) If $\Delta_1 \geq 0$ and $\Delta_2 \geq 0$, then $Z(T^*) = Z(T_2^*)$ and $T^* = T_2^*$.

(B) If $\Delta_1 > \Delta_2$, then

(B1) If $\Delta_1 < 0$ and $\Delta_2 < 0$, then $Z(T^*) = \min\{Z(M_1), Z(T_1^*)\}$ and $T^* = T_1^*$ or $M_1$ associated with the least cost.

(B2) If $\Delta_1 < 0$ and $\Delta_2 \geq 0$, then $Z(T^*) = Z(M_1)$ and $T^* = M_1$.

(B3) If $\Delta_1 \geq 0$ and $\Delta_2 \geq 0$, then $Z(T^*) = Z(T_2^*)$ and $T^* = T_2^*$.

**Proof.** See Appendix B.

**Theorem 3.** Suppose that the payment is paid at time $M_2$ and $\tilde{W}_3 > M_2$.

(A) If $\Delta_3 \leq \Delta_4$, then

(A1) If $\Delta_3 < 0$, $\Delta_3^* < 0$ and $\Delta_4 < 0$, then $Z(T^*) = \min\{Z(M_2), Z(T_3^*)\}$ and $T^* = M_2$ or $T_3^*$ associated with the least cost.

(A2) If $\Delta_3 < 0$, $\Delta_3^* \geq 0$ and $\Delta_4 < 0$, then $Z(T^*) = \min\{Z(M_2), Z(\tilde{W}_3)\}$ and $T^* = M_2$ or $\tilde{W}_3$ associated with the least cost.
(A3) If $\Delta_3 < 0$, $\Delta_3^* < 0$ and $\Delta_4 \geq 0$, then $Z(T^*) = \min\{Z(T_4^*), Z(\bar{T}_3^*)\}$ and $T^* = T_4^*$ or $T_3^*$ associated with the least cost.

(A4) If $\Delta_3 < 0$, $\Delta_3^* \geq 0$ and $\Delta_4 \geq 0$, then $Z(T^*) = \min\{Z(T_4^*), Z(\bar{T}_3^*)\}$ and $T^* = T_4^*$ or $\bar{T}_3^*$ associated with the least cost.

(A5) If $\Delta_3 \geq 0$, $\Delta_3^* \geq 0$ and $\Delta_4 \geq 0$, then $Z(T^*) = Z(T_4^*)$ and $T^* = T_4^*$.

(B) If $\Delta_3 > \Delta_4$, then

(B1) If $\Delta_3 < 0$, $\Delta_3^* < 0$ and $\Delta_4 < 0$, then $Z(T^*) = \min\{Z(M_2), Z(T_3^*)\}$ and $T^* = M_2$ or $T_3^*$ associated with the least cost.

(B2) If $\Delta_3 < 0$, $\Delta_3^* \geq 0$ and $\Delta_4 < 0$, then $Z(T^*) = \min\{Z(M_2), Z(\bar{T}_3^*)\}$ and $T^* = M_2$ or $\bar{T}_3^*$ associated with the least cost.

(B3) If $\Delta_3 \geq 0$, $\Delta_3^* \geq 0$ and $\Delta_4 < 0$, then $Z(T^*) = Z(T_4^*)$ and $T^* = T_4^*$.

(B4) If $\Delta_3 \geq 0$, $\Delta_3^* \geq 0$ and $\Delta_4 \geq 0$, then $Z(T^*) = Z(M_2)$ and $T^* = M_2$.

Theorem 4. Suppose that the payment is paid at time $M_2$ and $\bar{T}_3 \leq M_2$.

(A) If $\Delta_3 \leq \Delta_4$, then

(A1) If $\Delta_3 < 0$ and $\Delta_4 < 0$, then $Z(T^*) = \min\{Z(M_2), Z(T_3^*)\}$ and $T^* = M_2$ or $T_3^*$ associated with the least cost.

(A2) If $\Delta_3 < 0$ and $\Delta_4 \geq 0$, then $Z(T^*) = Z(T_3^*)$ and $T^* = T_3^*$.

(A3) If $\Delta_3 \geq 0$ and $\Delta_4 \geq 0$, then $Z(T^*) = \min\{Z(T_4^*), Z(T_3^*)\}$ and $T^* = T_4^*$ or $T_3^*$ associated with the least cost.

(B) If $\Delta_3 > \Delta_4$, then

(B1) If $\Delta_3 < 0$ and $\Delta_4 < 0$, then $Z(T^*) = \min\{Z(M_2), Z(T_3^*)\}$ and $T^* = T_3^*$ or $M_2$ associated with the least cost.

(B2) If $\Delta_3 < 0$ and $\Delta_4 \geq 0$, then $Z(T^*) = Z(M_2)$ and $T^* = M_2$.

(B3) If $\Delta_3 \geq 0$ and $\Delta_4 < 0$, then $Z(T^*) = Z(T_4^*)$ and $T^* = T_4^*$.

Proof. It is similar to the proof of Theorems 1 and 2, respectively.

Although $Z_i(T)$ ($i = 1, 2, 3$ and 4) are convex with respect to $T > 0$, it is appropriate to use many numerical methods such as the Newton–Raphson method which can be used to locate the optimal ordering time numerically. However, the solution procedure by the Newton–Raphson method suffers from the differential calculation, for this reason, it may not be easy for a practitioner with limited mathematical knowledge to understand the method. Consequently, there exists a motivation to find a simple, practical and accurate algorithm to determine the optimal ordering time. In doing this, we will provide bounds for searching the optimal replenishment time which minimizes the total variable cost per unit time.

Lemma 4. Suppose that $\Delta_i < 0$, then $M_i < T_i^* < T_i^a$ for $i = 1, 3$ where

$$T_i^a = \left[\theta S + \frac{(Dh + B_i \theta^2)}{\theta} + \frac{\theta I_i}{2pD} (2B_i^2 + W_i^2)\right] / (Dh + B_i \theta^2).$$

Proof. Since $\Delta_i < 0$, we have $Z_i'(M_i) < 0$. Moreover, $pD M_i + pI_i D M_i^2 / 2 < c(1 - r_i) I(0)$, then we have $W_i < B_i e^{\theta T}$, so

$$Z_i'(T) = -\frac{S}{T^2} + \frac{Dh + B_i \theta^2}{\theta^2 T^2} (\theta T e^{\theta T} - e^{\theta T} + 1) + \frac{pI_i D M_i^2}{2T^2}$$

$$+ \frac{I_i}{2pDT^2}[2B_i^2 \theta T e^{2\theta T} - B_i^2 e^{2\theta T} - 2B_i W_i \theta T e^{\theta T} + 2B_i W_i e^{\theta T} - W_i^2]$$

$$> -\frac{S}{T^2} + \frac{Dh + B_i \theta^2}{\theta^2 T^2} e^{\theta T} (\theta T - 1) + e^{-\theta T} + \frac{I_i}{2pDT^2}[2B_i^2 \theta T e^{2\theta T} - B_i^2 e^{2\theta T} - 2B_i W_i \theta T e^{\theta T}$$

$$+ 2B_i W_i e^{\theta T} - W_i^2]$$

Please cite this article in press as: K.-N. Huang, J.-J. Liao, A simple method to locate the optimal solution for exponentially deteriorating items under trade credit financing, Computers and Mathematics with Applications (2008), doi:10.1016/j.camwa.2007.08.049
Since and Lemmas 4

\[ Z(\theta T - 1) - \frac{I_c}{2pDT^2}[2(1 - \theta T)(B_i e^{\theta T} - W_i)B_i e^{\theta T} + W_i^2] \]

\[ = e^{\theta T} \left\{ -Se^{-\theta T} + \frac{Dh + B_i \theta^2}{\theta^2} (\theta T - 1) - \frac{I_c}{2pD} [2(1 - \theta T)(B_i e^{\theta T} - W_i)B_i + W_i^2 e^{-\theta T}] \right\} \]

\[ > e^{\theta T} \left\{ -S + \frac{Dh + B_i \theta^2}{\theta^2} (\theta T - 1) - \frac{I_c}{2pD} [2e^{-\theta T}(B_i e^{\theta T} - W_i)B_i + W_i^2 e^{-\theta T}] \right\} \]

\[ > e^{\theta T} \left\{ -S + \frac{Dh + B_i \theta^2}{\theta^2} (\theta T - 1) - \frac{I_c}{2pD} (2B_i^2 + W_i^2) \right\}. \]

Hence \( Z'(T_i^m) > 0 = Z'(T_i^e). \) Since \( Z'(T) \) is increasing on \((0, \infty)\) and \( Z'(T_i^m) > Z'(T_i^e) > Z'(\tilde{M}_i) \), we obtain \( \tilde{M}_i < T_i^e < T_i^m. \) Consequently, we have completed the proof. ☐

**Lemma 5.** Suppose that \( \Delta_j > 0 \), then \( T_j^l < T_j^e < \tilde{M}_j \) for \( j = 2, 4 \) where

\[ T_j^l = -\langle \theta S \rangle + \sqrt{\langle (\theta S)^2 + 2S[D(h + pI_d) + B_j\theta^2]/[D(h + pI_d) + B_j\theta^2]}. \]

**Proof.** Since \( \Delta_j > 0 \), we have \( Z'(\tilde{M}_j) > 0 \). Now, we get

\[ Z'(T) = \frac{S}{T^2} + \frac{Dh + B_i \theta^2}{\theta^2 T^2} (\theta T e^{\theta T} - e^{\theta T} + 1) + \frac{pI_d D}{2} \]

\[ < e^{\theta T} \left\{ -Se^{-\theta T} + \frac{Dh + B_i \theta^2}{\theta^2} \left( \frac{\theta^2 T^2}{2} \right) + \frac{pI_d DT^2}{2} e^{-\theta T} \right\} \]

\[ < e^{\theta T} \left\{ -S(1 - \theta T) + \frac{Dh + B_i \theta^2}{\theta^2} \left( \frac{\theta^2 T^2}{2} \right) + \frac{pI_d DT^2}{2} \right\} \]

\[ = e^{\theta T} \left\{ -S + (\theta S)T + \frac{D(h + pI_d) + B_j\theta^2}{2} (\theta T) \right\}. \]

Hence \( Z'(T_j^e) < 0 = Z'(T_j^m). \) Since \( Z'(T) \) is increasing on \((0, \infty)\) and \( Z'(T_j^m) < Z'(T_j^e) < Z'(\tilde{M}_j) \), we obtain \( \tilde{M}_j > T_j^e > T_j^m. \) Consequently, we have completed the proof. ☐

Based on Lemmas 4 and 5, there is an algorithm to find locate \( T_i^e \) \((i = 13)\) and \( T_j^e \) \((j = 24)\) can be developed.

The algorithm to find \( T_i^e \) \((i = 13)\)

**Step 1:** Let \( \varepsilon > 0. \)

**Step 2:** Let \( t_U = T_i^m \) and \( t_L = \tilde{M}_i. \)

**Step 3:** Let

\[ t_{opt} = \frac{t_L + t_U}{2}. \]

**Step 4:** If \( |Z'(t_{opt})| < \varepsilon \), go to Step 6. Otherwise, go to Step 5.

**Step 5:** If \( Z'(t_{opt}) > 0 \), set \( t_U = t_{opt} \). If \( Z'(t_{opt}) < 0 \), set \( t_L = t_{opt} \). Then go to Step 3.

**Step 6:** \( T_i^e = t_{opt} \) and exit the optimal replenishment time.

The algorithm to find \( T_j^e \) \((j = 24)\)

**Step 1:** Let \( \varepsilon > 0. \)

**Step 2:** Let \( t_U = \tilde{M}_j \) and \( t_L = T_j^l. \)

**Step 3:** Let

\[ t_{opt} = \frac{t_L + t_U}{2}. \]

**Step 4:** If \( |Z'(t_{opt})| < \varepsilon \), go to Step 6. Otherwise, go to Step 5.

**Step 5:** If \( Z'(t_{opt}) > 0 \), set \( t_U = t_{opt} \). If \( Z'(t_{opt}) < 0 \), set \( t_L = t_{opt} \). Then go to Step 3.

**Step 6:** \( T_j^e = t_{opt} \) and exit the optimal replenishment time.
it is not difficult to see that when

Table 1

Relative errors for Problem 1

<table>
<thead>
<tr>
<th>Relative errors</th>
<th>( \theta ) values</th>
<th>0.03</th>
<th>0.1</th>
<th>0.5</th>
<th>0.75</th>
<th>1</th>
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<tbody>
<tr>
<td>( D ) values</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10,000</td>
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<td>-0.0136</td>
<td>0.0191</td>
<td>0.0085</td>
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<tr>
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<td></td>
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<td>-0.0205</td>
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</tr>
<tr>
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</tr>
<tr>
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<tr>
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<td>-0.5503</td>
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<td>60,000</td>
<td></td>
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<td>-0.5433</td>
<td>-0.5895</td>
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</table>

Table 2

Relative errors for Problem 2

<table>
<thead>
<tr>
<th>( I_d )</th>
<th>Chang and Teng’s solutions</th>
<th>Our solutions</th>
<th>Percent improvement ( I_d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_C^* )</td>
<td>( Q(T_C^*) )</td>
<td>( T_{HL}^* )</td>
<td>( Q(T_{HL}^*) )</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
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<td>0.054709</td>
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<td>52.9974</td>
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<td>0.048559</td>
<td>48.5940</td>
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</tr>
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</table>

4. The problem in a numerical example

Let \( T_C^* \) and \( T_{HL}^* \) denote the solutions obtained by Chang and Teng [1] and solutions obtained by this paper, respectively, for a pair of values of and \( \theta \) and \( D \). A relative error for a given pair of \( \theta \) and \( D \) is defined as follows:

\[
\varepsilon = \frac{Q(T_C^*) - Q(T_{HL}^*)}{Q(T_{HL}^*)}.
\]

Problem 1. Consider that \( S = $60, \ h = $4/\text{unit/year}, \ I_c = 0.2/\text{year}, \ I_d = 0.12/\text{year}, \ r = 0.1, \ M_1 = 0.3, \ c = \ $100 \text{ per unit}, \ p = $200 \text{ per unit.} \)

From Table 1 it is not difficult to see that when \( \theta \) is small, the solution obtained from Chang and Teng [1] work well regardless of the magnitude of \( D \). On the other hand, a large value of \( \theta \) implies that in all the cases considered our optimization procedure yields higher costs than that of Chang and Teng [1].

Problem 2. Consider that \( S = $10, \ \theta = 0.03, \ h = $4/\text{unit/year}, \ I_c = 0.09/\text{year}, \ c = $20 \text{ per unit,} \ p = $30 \text{ per unit,} \ D = 1000 \text{ unit/year,} \ r = 0.02 \text{ and} \ M_1 = 20/365 \text{ year.} \)

In Table 2, when \( \theta \) is small, the solution obtained from Chang and Teng [1] work well as well. However, in some situations, it may be inadequate that Chang and Teng [1] take \( Q(T_C^*) \) as the optimal order quantity. Consequently, our optimization procedure performs relatively well.

5. Sensitivity analysis

The following numerical example is considered to study the effects of changing parameter values on the optimal replenishment time and on the optimal total cost per unit time. Whenever one parameter is changed by some percentage, all other parameters remain at their original values. The results obtained are summarized in Tables 3–8.

Example 1. In this example, we study the effects of the changes in the model parameter \( S \) on the optimal solutions. Given \( D = 1000 \text{ unit/year}, \ \theta = 0.03, \ h = $4/\text{unit/year}, \ I_d = 0.06/\text{year}, \ I_c = 0.09/\text{year}, \ c = $20 \text{ per unit,} \ p = $30 \text{ per unit.} \)
Table 3
Optimal replenishment solution with changing parameter $S$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$\Delta_1$</th>
<th>$\Delta^*_1$</th>
<th>$\Delta_2$</th>
<th>$T^*$</th>
<th>$Q^*$</th>
<th>$Z(T^*)$</th>
</tr>
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<tr>
<td>10</td>
<td>$&lt;0$</td>
<td>$&gt;0$</td>
<td>$&lt;0$</td>
<td>0.0548</td>
<td>4.8351</td>
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<tr>
<td>20</td>
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<td>$&lt;0$</td>
<td>$&lt;0$</td>
<td>0.0862</td>
<td>86.3061</td>
<td>19.999</td>
</tr>
<tr>
<td>30</td>
<td>$&lt;0$</td>
<td>$&lt;0$</td>
<td>$&lt;0$</td>
<td>0.1040</td>
<td>104.5582</td>
<td>20.104</td>
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Table 4
Optimal replenishment solution with changing parameter $S$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$\Delta_3$</th>
<th>$\Delta^*_3$</th>
<th>$\Delta_4$</th>
<th>$T^*$</th>
<th>$Q^*$</th>
<th>$Z(T^*)$</th>
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<td>$&gt;0$</td>
<td>$&gt;0$</td>
<td>0.0559</td>
<td>55.9269</td>
<td>20.210</td>
</tr>
<tr>
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<td>$&gt;0$</td>
<td>$&gt;0$</td>
<td>0.0790</td>
<td>79.1037</td>
<td>20.358</td>
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<tr>
<td>30</td>
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<td>$&gt;0$</td>
<td>$&lt;0$</td>
<td>0.0822</td>
<td>106.8627</td>
<td>20.471</td>
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</table>

Table 5
Optimal replenishment solution with changing parameter $I_d$

<table>
<thead>
<tr>
<th>$I_d$</th>
<th>$\Delta_1$</th>
<th>$\Delta^*_1$</th>
<th>$\Delta_2$</th>
<th>$T^*$</th>
<th>$Q^*$</th>
<th>$Z(T^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>$&lt;0$</td>
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<td>$&lt;0$</td>
<td>0.0548</td>
<td>54.8351</td>
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<td>0.0535</td>
<td>53.5229</td>
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Table 6
Optimal replenishment solution with changing parameter $I_d$

<table>
<thead>
<tr>
<th>$I_d$</th>
<th>$\Delta_3$</th>
<th>$\Delta^*_3$</th>
<th>$\Delta_4$</th>
<th>$T^*$</th>
<th>$Q^*$</th>
<th>$Z(T^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.06</td>
<td>$&gt;0$</td>
<td>$&gt;0$</td>
<td>$&gt;0$</td>
<td>0.0559</td>
<td>55.9269</td>
<td>20.210</td>
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<tr>
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<td>$&gt;0$</td>
<td>$&gt;0$</td>
<td>$&gt;0$</td>
<td>0.0546</td>
<td>54.6648</td>
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<td>$&gt;0$</td>
<td>0.0534</td>
<td>53.4728</td>
<td>20.177</td>
</tr>
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</table>

1 (A) When $M_1 = 20/365$ year = 0.0548 year, $W_1 = 0.0839$ and $r = 0.02$.

2 (B) When $M_2 = 30/365$ year = 0.0822 year and $W_3 = 0.1234$.

**Example 2.** In this example, we study the effects of the changes in the model parameter $I_d$ on the optimal solutions.

Given $D = 1000$ unit/year, $h = $ 4/unit/year, $I_c = 0.09$/year, $\theta = 0.03$, $S = 10$, $c = $ 20 per unit, $p = $ 30 per unit.

6 (A) When $M_1 = 20/365$ year = 0.0548 year, $W_1 = 0.0839$ and $r = 0.02$.

7 (B) When $M_2 = 30/365$ year = 0.0822 year and $W_3 = 0.1234$.

**Example 3.** In this example, we study the effects of the changes in the model parameters $p$ and $\theta$ on the optimal solutions. Given $D = 1000$ unit/year, $h = $ 4/unit/year, $I_d = 0.06$/year, $I_c = 0.09$/year, $\theta = 0.03$, $S = 10$ and $c = $ 20 per unit.

11 (A) When $M_1 = 20/365$ year = 0.0548 year, $W_1 = 0.0839$ and $r = 0.02$

12 (B) When $M_2 = 30/365$ year = 0.0822 year and $W_3 = 0.1234$. The following inferences can be obtained from the sensitivity analysis based on Tables 3–8.

14 (1) Tables 3 and 4 show that as the value of $S$ increases, $T^*$, $Q^*$ and $Z(T^*)$ increase when the supplier offers a cash discount or a permissible delay to the customer.

15 (2) Tables 5 and 6 show that as the value of $I_d$ increases, $T^*$, $Q^*$ and $Z(T^*)$ decrease when the supplier offers a cash discount or a permissible delay to the customer.

Please cite this article in press as: K.-N. Huang, J.-J. Liao, A simple method to locate the optimal solution for exponentially deteriorating items under trade credit financing, Computers and Mathematics with Applications (2008), doi:10.1016/j.camwa.2007.08.049
Theorems 1–4 and Tables 7 reveal which one of \( Z_1(T) \), \( Z_2(T) \) and \( Z_3(T) \) is optimal. An efficient algorithm procedure is provided to locate the optimal ordering time. Basically, if production managers use the solution algorithm in this paper, they will find it simple, rapid and accurate. From the point of implementation, the solution algorithm in this paper is rather beneficial. Finally, the sensitivity of the solution to the values of different parameters has been discussed. In further analysis, for the completeness of the discussion, we would like to consider the case that \( pD + pLdDM_i^2/2 \geq c(1 - \bar{r}_i)I(0) \) for \( i = 1, 3 \) and \( pD\bar{M}_j + pLdDM_j^2/2 \geq cI(0) \) for \( j = 2, 4 \) in the inventory model.

Appendix A

(A1) If \( \Delta_1 < 0 \) and \( \Delta_2 < 0 \), then \( Z'_1(M_1) < 0 \) and \( Z'_2(M_1) < 0 \) which imply that \( T_1^* > M_1 \) and \( T_2^* > M_1 \).

Furthermore, \( Z_1(T) \) has the minimum value at \( T_1^* \) when \( T > M_1 \) and \( Z_2(T) \) has the minimum value at \( M_1 \).
when \( T \leq M_1 \). Since \( \Delta_1 > 0 \) which implies that \( T_1^* > \bar{W}_1 \), hence \( Z(T^*) = \min\{Z(M_1), Z(T_1^*)\} \) and \( T^* = M_1 \) or \( T_1^* \) associated with the least cost.

(A2) If \( \Delta_1 < 0 \) and \( \Delta_2 < 0 \), then \( Z_1'(M_1) < 0 \) and \( Z_2'(M_1) < 0 \) which imply that \( T_1^* > M_1 \) and \( T_2^* > M_1 \). Furthermore, \( Z_1(T) \) has the minimum value at \( T_1^* \) when \( T > M_1 \) and \( Z_2(T) \) has the minimum value at \( M_1 \) when \( T \leq M_1 \). Since \( \Delta_1 > 0 \) which implies that \( T_1^* \leq \bar{W}_1 \), hence \( Z(T^*) = \min\{Z(T_1^*), Z(\bar{W}_1)\} \) and \( T^* = M_1 \) or \( \bar{W}_1 \) associated with the least cost.

(A3) If \( \Delta_1 < 0 \) and \( \Delta_2 \geq 0 \), then \( Z_1'(M_1) < 0 \) and \( Z_2'(M_1) \geq 0 \) which imply that \( T_1^* > M_1 \) and \( T_2^* \leq M_1 \). Furthermore, \( Z_1(T) \) has the minimum value at \( T_1^* \) when \( T > M_1 \) and \( Z_2(T) \) has the minimum value at \( T_2^* \) when \( T \leq M_1 \). Since \( \Delta_1 > 0 \) which implies that \( T_1^* \leq \bar{W}_1 \), hence \( Z(T^*) = \min\{Z(T_2^*), Z(\bar{W}_1)\} \) and \( T^* = T_2^* \) or \( \bar{W}_1 \) associated with the least cost.

(A4) If \( \Delta_1 < 0 \) and \( \Delta_2 \geq 0 \), then \( Z_1'(M_1) < 0 \) and \( Z_2'(M_1) \geq 0 \) which imply that \( T_1^* > M_1 \) and \( T_2^* \leq M_1 \). Furthermore, \( Z_1(T) \) has the minimum value at \( T_1^* \) when \( T > M_1 \) and \( Z_2(T) \) has the minimum value at \( T_2^* \) when \( T \leq M_1 \). Since \( \Delta_1 > 0 \) which implies that \( T_1^* \leq \bar{W}_1 \), and we have \( Z_1(\bar{W}_1) > Z_1(M_1) > Z_2(M_1) > Z_2(T_2^*) \). Hence \( Z(T^*) = Z(T_2^*) \) and \( T^* = T_2^* \).

(B1), (B2) and (B3) is similar to the proof of (A1), (A2) and (A5), respectively.

(B4) If \( \Delta_1 \geq 0 \) and \( \Delta_2 < 0 \), then \( Z_1'(M_1) \geq 0 \) and \( Z_2'(M_1) < 0 \) which imply that \( T_1^* \leq M_1 \) and \( T_2^* > M_1 \). Furthermore, \( Z_1(T) \) has the minimum value at \( M_1 \) when \( T \geq M_1 \) and \( Z_2(T) \) has the minimum value at \( T_2^* \) when \( T \leq M_1 \). Since \( \Delta_1 \geq 0 \) which implies that \( T_1^* \leq \bar{W}_1 \), and we have \( Z_1(\bar{W}_1) > Z_1(M_1) > Z_2(M_1) \). Hence \( Z(T^*) = Z(M_1) \) and \( T^* = M_1 \).

We have completed the proof of Theorem 1. \( \square \)

Appendix B

(A1) If \( \Delta_1 < 0 \) and \( \Delta_2 < 0 \), then \( Z_1'(M_1) < 0 \) and \( Z_2'(M_1) < 0 \) which imply that \( T_1^* > M_1 \) and \( T_2^* > M_1 \). Furthermore, \( Z_1(T) \) has the minimum value at \( T_1^* \) when \( T \geq M_1 \) and \( Z_2(T) \) has the minimum value at \( M_1 \) when \( T \leq M_1 \). Hence \( Z(T^*) = \min\{Z(T_1^*), Z(M_1)\} \) and \( T^* \) is \( T_1^* \) or \( M_1 \) associated with the least cost.

(A2) If \( \Delta_1 < 0 \) and \( \Delta_2 \geq 0 \), then \( Z_1'(M_1) < 0 \) and \( Z_2'(M_1) \geq 0 \) which imply that \( T_1^* > M_1 \) and \( T_2^* \leq M_1 \). Furthermore, \( Z_1(T) \) has the minimum value at \( T_1^* \) when \( T \geq M_1 \) and \( Z_2(T) \) has the minimum value at \( T_2^* \) when \( T \leq M_1 \). Hence \( Z(T^*) = \min\{Z(T_1^*), Z(T_2^*)\} \) and \( T^* \) is \( T_1^* \) or \( T_2^* \) associated with the least cost.

(A3) If \( \Delta_1 \geq 0 \) and \( \Delta_2 \geq 0 \), then \( Z_1'(M_1) \geq 0 \) and \( Z_2'(M_1) \geq 0 \) which imply that \( T_1^* \leq M_1 \) and \( T_2^* \leq M_1 \). Furthermore, \( Z_1(T) \) has the minimum value at \( M_1 \) when \( T \geq M_1 \) and \( Z_2(T) \) has the minimum value at \( T_2^* \) when \( T \leq M_1 \). Since \( Z_2(T_2^*) < Z_2(M_1) < Z_1(M_1) \), and we have \( Z(T^*) = Z(T_2^*) \) and \( T^* = T_2^* \).

(B1) and (B3) is similar to the proof of (A1) and (A3), respectively.

(B2) If \( \Delta_1 \geq 0 \) and \( \Delta_2 < 0 \), then \( Z_1'(M) \geq 0 \) and \( Z_2'(M) < 0 \) which imply that \( T_1^* \leq M_1 \) and \( T_2^* > M_1 \). Furthermore, \( Z_1(T) \) has the minimum value at \( M_1 \) when \( T \geq M_1 \) and \( Z_2(T) \) has the minimum value at \( M_1 \) when \( T \leq M_1 \). Since \( Z_1(M_1) > Z_2(M_1) \), then we have \( Z(T^*) = Z(M_1) \) and \( T^* = M_1 \).

We have completed the proof of Theorem 2. \( \square \)

References


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