Minimax distribution free procedure with backorder price discount

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Received 24 September 2005; accepted 23 November 2006
Available online 13 January 2007

Abstract

The inventory models analyzed in this paper explore the problem in which the lead time and ordering cost reductions are inter-dependent in the continuous review inventory model with backorder price discount. The objective is to minimize total related cost by simultaneously optimizing the order quantity, reorder point, lead time and backorder price discount. Moreover, we assume that the mean and variance of the lead time demand are known, but its probability distribution is unknown. We apply a minimax distribution free procedure to find the optimal solution, and three numerical examples are given to illustrate the results.

Keywords: Backorder price discount; Lead time; Minimax distribution free procedure

1. Introduction

In classical economic order quantity (EOQ) model dealing with inventory problems, either using deterministic or probabilistic models, lead time is viewed as a prescribed constant or a stochastic variable. Therefore, lead time is not subject to control (see, e.g., Naddor, 1966; Johnson and Montgomery, 1974; Silver and Peterson, 1985). However, this may not be realistic. In many practical situations, lead time can be shortened at an added crashing cost; in other words, it is controllable. By shortening the lead time, we can lower the safety stock, reduce the stockout loss and improve the service level to the customer so as to increase the competitive edge in business.

Recently, several authors have presented various inventory models with lead time reduction. Initially, Liao and Shyu (1991) presented an inventory model in which the lead time is a unique decision variable and the order quantity is predetermined. Ben-Daya and Raouf (1994) extended Liao and Shyu’s (1991) model by allowing both the lead time and the order quantity as decision variables. Ouyang et al. (1996) generalized Ben-Daya and Raouf’s (1994) model and considered shortages with partial backorders, while Pan and Hsiao (2001) revised Ouyang et al.’s (1996) model to consider the backorder price discount as one of the decision variables.

It is noticed that the above papers Liao and Shyu (1991), Ben-Daya and Raouf (1994), Ouyang et al. (1996), Pan and Hsiao (2001) are all focused on the continuous review inventory model to derive the benefits from lead time reduction, and the ordering cost is treated as a fixed constant. In a recent paper,
Ouyang et al. (1999) proposed two continuous review inventory models to study the effects of lead time and ordering cost reductions. We note that the lead time and ordering cost reductions in Ouyang et al. (1999) are assumed to act independently. However, this is only one of the possible cases. In practice, the lead time and ordering cost reductions may be related closely; the reduction of lead time may accompany the reduction of ordering cost, and vice versa. For example, according to Silver and Peterson (1985, p. 150), the implementation of electronic data interchange (EDI) may reduce the lead time and ordering cost simultaneously. Therefore, it is more reasonable to assume that the ordering cost reduction vary according to different lead times.

In the real market, as unsatisfied demands occur, we can often observe that some customers may prefer their demands to be backordered, and some may refuse the backorder case. There is a potential factor that may motivate the customers’ desire for backorders. The factor is an offering of a backorder price discount from a supplier. Pan and Hsiao (2001). In general, provided that a supplier could offer a backorder price discount on the stockout item by negotiation to secure more backorders, it may make the customers more willing to wait for the desired items. In other words, the bigger the backorder price discount, the bigger the advantage to the customers, and hence, a larger number of backorder ratio may result. This phenomenon reveals that, as unsatisfied demands occur during the stockout period, how to find an optimal backorder ratio through controlling a backorder price discount from a supplier to minimize the relevant inventory total cost is a decision-making problem worth discussing.

In this paper, we attempt to modify Pan and Hsiao’s (2001) model for a minimax distribution free inventory model that includes a controllable backorder price discount and the reduction of lead time accompanies a decrease of ordering cost. For this case, we solve the problem by using the minimax distribution free approach, which was originally proposed by Scarf (1958). Recently, Gallego and Moon (1993) presented a new and very compact proof of the optimality of Scarf’s (1958) ordering rule. Also, Hariga and Ben-Daya (1999), Moon and Choi (1995, 1997), Moon and Silver (2000), Ouyang and Wu (1998), Ouyang and Chang (2002), Ouyang et al. (2004), Silver and Moon (2001) applied this approach to some production/inventory models. Moreover, note that the previous works on distribution free approach and partial lost sales (or backorders) are well documented in Silver et al. (1998). In this study, the objective is to minimize the total related cost by optimizing the order quantity, reorder point, backorder price discount and lead time, simultaneously. Furthermore, the effects of parameters are also included and three illustrative numerical examples are given.

2. Notation and assumptions

The mathematical models in this paper are developed on the basis of the following notation and assumptions.

**Notation**

- \( D \) = average demand per year
- \( A_0 \) = original ordering cost (before any investment is made)
- \( A \) = ordering cost per order, \( 0<A\leq A_0 \)
- \( h \) = inventory holding cost per unit per year
- \( Q \) = order quantity (a decision variable)
- \( r \) = reorder point (a decision variable)
- \( \beta \) = fraction of the shortage that will be backordered, i.e., backorder ratio, \( 0<\beta<1 \)
- \( \beta_0 \) = upper bound of the backorder ratio
- \( \pi_s \) = backorder price discount offered by the supplier per unit (a decision variable)
- \( \pi_o \) = marginal profit (i.e., cost of lost demand per unit)
- \( L \) = length of lead time (a decision variable)
- \( X \) = lead time demand
- \( f_X(x) \) = the probability density function (p.d.f.) of \( X \) with finite mean \( DL \) and standard deviation \( \sigma \sqrt{L} \), where \( \sigma \) denotes the standard deviation of the demand per unit time
- \( E(\cdot) \) = mathematical expectation
- \( x^+ \) = maximum value of \( x \) and 0, i.e., \( x^+ = \text{Max}\{x,0\} \).

**Assumptions**

1. The reorder point \( r \) = expected demand during lead time + safety stock (SS), and \( SS = k \times (\text{standard deviation of lead time demand}) \), i.e., \( r = DL + k\sigma \sqrt{L} \), where \( k \) is the safety factor.
2. Inventory is continuously reviewed. Replenishments are made whenever the inventory level falls to the reorder point \( r \).
(3) The lead time $L$ consists of $n$ mutually independent components. The $i$-th component has a minimum duration $a_i$ and normal duration $b_i$, and a crashing cost per unit time $c_i$. Further, for convenience, we rearrange $c_i$ such that $C_1 \leq C_2 \leq \cdots \leq C_n$. Then, it is clear that the reduction of lead time should be first on component 1 because it has the minimum unit crashing cost, and then component 2, and so on.

(4) If we let $L_0 = \sum_{j=1}^{n} b_j$ and $L_i$ be the length of lead time with components $1, 2, \ldots, i$ crashed to their minimum duration, then $L_i$ can be expressed as $L_i = \sum_{j=1}^{n} b_j - \sum_{j=1}^{i} (b_j - a_j)$, $i = 1, 2, \ldots, n$; and the lead time crashing cost, $C(L)$ per cycle for a given $L \in [L_i, L_{i-1}]$ is given by

$$C(L) = c_i(L_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j).$$

(5) The reduction of lead time $L$ accompanies a decrease of ordering cost $A$, and $A$ is a strictly concave function of $L$, i.e., $A'(L) > 0$ and $A''(L) < 0$.

(6) The supplier makes decisions in order to obtain profits. Therefore, if the price discount, $\pi_x$, is greater than the marginal profit, $\pi_0$, then the supplier may decide against offering the price discount.

(7) During the stockout period, the backorder ratio, $\beta$, is variable and is in proportion to the backorder price discount, $\pi_x$, offered by the supplier per unit. Thus, $\beta = \frac{\beta_0 \pi_x}{\pi_0}$, where $1 \leq \beta_0 < 1$ and $0 \leq \pi_x \leq \pi_0$ (see Pan and Hsiao, 2001).

3. The basic model

Recently, Pan and Hsiao (2001) considered a continuous review inventory model with backorder price discount and variable lead time. They assumed that the ordering cost is treated as a fixed constant and independent of lead time, and the lead time demand follows a normal distribution. In practice, information about the form of the probability distribution of the lead time demand is often limited. Thus, the normal distribution may not provide the best protection against the occurrence of the other distribution. Therefore, in this study, we will closely follow Pan and Hsiao (2001), and use a minimax distribution free procedure for the backorder price discount. We further consider the ordering cost reduction dependent on the lead time reduction. As mentioned earlier, we have assumed that the lead time demand $X$ has a p.d.f. $f_X(x)$ with finite mean $DL$ and standard deviation $\sigma/\sqrt{L}$, and the reorder point $r = DL + k\sigma/\sqrt{L}$. Specifically, by assumptions 1–5, the total expected annual cost, which is composed of ordering cost, inventory holding cost, stockout cost and lead time crashing cost, is expressed by

$$EAC(Q, r, \beta, L) = \frac{A(L)D}{Q} + h\left[\frac{Q}{2} + r - DL + (1 - \beta)E(X - r)^+\right] + \frac{D}{Q}(\pi_0 \pi_x + \pi_0 - \pi_0 \pi_x)E(X - r)^+ + \frac{D}{Q}C(L),$$

where $E(X - r)^+$ is the expected demand shortage at the end of cycle.

Moreover, by Assumption 7, during the stockout period, the backorder ratio $\beta$ is variable and is proportion to the backorder price discount offered by the supplier per unit, $\pi_x$, that is, $\beta = \frac{\beta_0 \pi_x}{\pi_0}$. Thus, the backorder price discount offered by the supplier per unit, $\pi_x$, can be treated as a decision variable instead of the backorder ratio, $\beta$. In this case, problem (1) can be transformed into the following formulation:

$$EAC(Q, r, \pi_x, L) = \frac{A(L)D}{Q} + h\left[\frac{Q}{2} + r - DL + \left(1 - \frac{\beta_0 \pi_x}{\pi_0}\right)E(X - r)^+\right] + \frac{D}{Q}\left(\frac{\beta_0 \pi_x^2}{\pi_0} + \pi_0 - \beta_0 \pi_x\right)E(X - r)^+ + \frac{D}{Q}C(L).$$

For the case that the distributional form of lead time demand $X$ is unknown, we can not find the exact value of $E(X - r)^+$. It is assumed that the probability distribution of lead time demand $X$ has given first two moments; i.e., the p.d.f. $f_X$ of $X$ belongs to the class $\Omega$ of p.d.f.’s with finite mean $DL$ and standard deviation $\sigma/\sqrt{L}$. Then, we propose to apply the minimax distribution free procedure for our problem. The minimax solution of an inventory problem has been first introduced by Scarf (1958). Later, Gallego and Moon (1993) presented a new and compact proof of the optimality of Scarf (1958). The minimax distribution free approach for this problem is to find the “most unfavorable” p.d.f. $f_X$ in $\Omega$ for $(Q, r, \pi_x, L)$ model, and then to minimize the total expected annual cost over $(Q, r, \pi_x, L)$. 


More exactly, we need to solve
\[
\min_{(Q, r, \pi_x, L)} \max_{f_X \in \Omega} \text{EAC}(Q, r, \pi_x, L). \tag{3}
\]

For this purpose, we need the following Proposition that was asserted by Gallego and Moon (1993).

**Proposition 1.** For any \( f_X \in \Omega \),
\[
E(X - r)^+ \leq \frac{1}{2} \sqrt{\sigma^2 L + (r - DL)^2} - (r - DL).
\]

(4)

Moreover, the upper bound (4) is tight.

Substituting \( r = DL + k\sigma\sqrt{L} \), \((k \text{ is the safety factor})\) into (4), we obtained the following inequality:
\[
E(X - r)^+ \leq \frac{1}{2} \sigma\sqrt{L} \left( \sqrt{1 + k^2} - k \right). \tag{5}
\]

For any probability distribution of the lead time \( X \), the above inequality (5) holds. And we can treated the safety factor \( k \) as a decision variable instead of reorder point \( r \). Therefore, using inequality (5) and model (2), problem (3) is reduced to minimize
\[
\text{EAC}^W(Q, k, \pi_x, L)
\]

\[
= \frac{A(L)D}{Q} + h \left[ \frac{Q}{2} + k\sigma\sqrt{L} \right]
\]

\[
+ \frac{1}{2} \left[ \frac{h}{1 - \frac{L_0\pi_x}{\pi_0}} + \frac{D}{Q} G(\pi_x) \right]
\]

\[
\times \sigma \sqrt{L} \left( \sqrt{1 + k^2} - k \right)
\]

\[
+ \frac{D}{Q} C(L),
\tag{6}
\]

where \( \text{EAC}^W(Q, k, \pi_x, L) \) is the least upper bound of \( \text{EAC}(Q, k, \pi_x, L) \) and \( G(\pi_x) = \pi_0 - \beta_0\pi_x + \frac{\beta_0\pi_x^2}{\pi_0} > 0 \) (because \( \pi_0/\pi_x > \beta_0(1)(\pi_x/\pi_0) > 0 \)).

It is clear that, for any \( k \), we have \( \sqrt{1 + k^2} - k > 0 \). Hence, for fixed \( (Q, k, \pi_x) \), \( \text{EAC}^W(Q, k, \pi_x, L) \) is concave in \( L \in [L_i, L_{i-1}] \), because
\[
\frac{\partial^2 \text{EAC}^W(Q, k, \pi_x, L)}{\partial L^2} = \frac{A'(L)D}{Q} - \frac{1}{4} h \sigma \sqrt{L}^{-3/2}
\]

\[
- \frac{1}{8} \left[ \frac{h}{1 - \frac{L_0\pi_x}{\pi_0}} + \frac{D}{Q} G(\pi_x) \right]
\]

\[
\times \sigma \sqrt{L}^{-3/2} \left( \sqrt{1 + k^2} - k \right) < 0.
\]

Therefore, for fixed \( (Q, k, \pi_x) \), the minimum total expected annual cost will occur at the end points of the interval \([L_i, L_{i-1}]\).

On the other hand, for fixed \( L \in [L_i, L_{i-1}] \), upon setting \( \partial \text{EAC}^W(Q, k, \pi_x, L) / \partial Q = 0 \), \( \partial \text{EAC}^W(Q, k, \pi_x, L) / \partial k = 0 \) and \( \partial \text{EAC}^W(Q, k, \pi_x, L) / \partial \pi_x = 0 \), we have
\[
0 = \frac{\partial \text{EAC}^W(Q, k, \pi_x, L)}{\partial Q}
\]

\[
= -\frac{A(L)D}{Q^2} + \frac{h}{2} \frac{DG(\pi_x)\sigma\sqrt{L}}{Q^2}
\]

\[
\times \left( \sqrt{1 + k^2} - k \right) - \frac{D}{Q} C(L),
\tag{7}
\]

and
\[
0 = \frac{\partial \text{EAC}^W(Q, k, \pi_x, L)}{\partial k}
\]

\[
= h\sigma\sqrt{L} + \frac{1}{2} \left[ \frac{h}{1 - \frac{L_0\pi_x}{\pi_0}} + \frac{D}{Q} G(\pi_x) \right]
\]

\[
\times \sigma\sqrt{L} \left( \frac{k}{\sqrt{1 + k^2}} - 1 \right),
\tag{8}
\]

From Eqs. (7) to (9), we obtain
\[
Q = \left[ \frac{2A(L)D + DG(\pi_x)\sigma\sqrt{L} (\sqrt{1 + k^2} - k) + 2DC(L)}{h} \right]^{1/2},
\tag{10}
\]

\[
1 - \frac{k}{\sqrt{1 + k^2}} = \frac{2h}{h (1 - \frac{L_0\pi_x}{\pi_0}) + \frac{DG(\pi_x)}{Q}}
\tag{11}
\]

and
\[
\pi_x = \frac{hQ}{2D} + \frac{\pi_0}{2}.
\tag{12}
\]

Theoretically, for fixed \( L \in [L_i, L_{i-1}] \) from Eqs. (10) to (12), we can get the values of \( Q, k \) and \( \pi_x \) (we denote these values by \( Q^*, k^* \) and \( \pi_x^* \)). Moreover, it can be shown that the second-order sufficient conditions are satisfied because the Hessian matrix is positive definite at point \((Q^*, k^*, \pi_x^*)\) (see the Appendix for the detail proof). Hence, for fixed \( L \in [L_i, L_{i-1}] \), the point \((Q^*, k^*, \pi_x^*)\) is the optimal...
solution such that the total expected annual cost has a minimum value.

An explicit general solution for \((Q^*, k^*, \pi^*_x)\) is not possible because the evaluation of each of Eqs. (10)–(12) requires a knowledge of the value of the other. Consequently, substituting Eq. (12) into Eqs. (10) and (11), we get

\[
Q = \left\{ \frac{2D}{h} \left[ A(L) + \frac{\pi_0 (4 - \beta_0)}{8} \sqrt{L} \left( \sqrt{1 + k^2} - k \right) + C(L) \right] \right\}^{1/2}
\]

and

\[
1 - \frac{k}{\sqrt{1 + k^2}} = \frac{8Qh\pi_0D}{(4 - 2\beta_0)Q\pi_0Dh + \pi_0^2D^2(4 - \beta_0) - \beta_0 h^2 Q^2}.
\]

Note that from (10) and \(G(\pi_x) > 0\), the order quantity \(Q\) is greater than zero.

From (13) to (14), though it is difficult to find the closed form solution of \((Q^*, k^*)\), the optimal value of \((Q^*, k^*)\) can be obtained by solving Eqs. (13) and (14) iteratively until convergence. Once the optimal solution of \((Q^*, k^*)\) is obtained, the optimal value of \(\pi^*_x\) can be determined from (12). Therefore, we can establish the following Algorithm to find the optimal values for the order quantity \(Q^*\), safety factor \(k^*\), backorder price discount \(\pi^*_x\) and lead time \(L^*\).

**Algorithm.** Step 1: For each \(L_i\), \(i = 0, 1, 2, ..., n\), perform (a)–(e).

(a) Start with \(k_{i1} = 0\).
(b) Substituting the value of \(k_{i1}\) into (13) to evaluate \(Q_{i1}\).
(c) Utilizing \(Q_{i1}\) determines \(k_{i2}\) from (14).
(d) Repeat (b) and (c) until no change occurs in the values of \(Q_i\) and \(k_i\).
(e) Substituting \(Q_i\) and \(k_i\) into (12) computes \(\pi_{x_i}\).

Step 2: Compare \(\pi_{x_i}\) and \(\pi_0\).

(a) If \(\pi_{x_i} \leq \pi_0\), \(\pi_{x_i}\) is feasible, then go to Step 3.
(b) If \(\pi_{x_i} > \pi_0\), \(\pi_{x_i}\) is not feasible. Set \(\pi_{x_i} = \pi_0\) and calculate the corresponding value of \(Q_i\) and \(k_i\) from Eqs. (10) and (11) iteratively until convergence (the solution procedure is similar to that given in Step 1), then go to Step 3.

Step 3: Compute the corresponding total expected annual cost

\[
EAC^W(Q_i, k_i, \pi_{x_i}, L_i) = \frac{A(L)D}{Q_i} + \frac{h}{2} \left[ 1 - \frac{\beta_0 \pi_{x_i}}{\pi_0} + \frac{D}{Q_i} G(\pi_{x_i}) \right] \times \sqrt{L_i} \left( 1 + k_i^2 - k_i \right).
\]

Step 4: Find \(\min_{i=0,1,2,...,n} EAC^W(Q_i, k_i, \pi_{x_i}, L_i)\). If \(EAC^W(Q^*, k^*, \pi^*_x, L^*) = \min_{i=0,1,2,...,n} EAC^W(Q_i, k_i, \pi_{x_i}, L_i)\), then \((Q^*, k^*, \pi^*_x, L^*)\) is the optimal solution.

Note that once \((Q^*, k^*, \pi^*_x, L^*)\)is obtained, the optimal reorder point \(r^* = DL^* + k^* \sigma \sqrt{L^*}\), the optimal backorder ratio \(\beta^* = \beta_0 \pi^*_x / \pi_0\) and the optimal ordering cost \(A^* = A(L^*)\) follows.

4. Numerical examples

The numerical examples given below illustrate the above solution procedure, we consider an inventory system with the following data used in Pan and Hsiao (2001): \(D = 600\) units per year, \(A_0 = 200\) per order, \(h = 20\) per unit per year, \(\pi_0 = 150\) per unit, \(\sigma = 7\) units per week, and the lead time has three components with data shown in Table 1.

**Example 1.** We assume that lead time and ordering cost reductions act dependently with the following relationship (Chen et al., 2001; Chiu, 1988): \(A_0 - A_i = (1/\lambda)(L_0 - L_i)\), which implies \(A(L) = a + bL\), where \(\lambda > 0\), \(a = (1 - 1/\lambda)A_0\) and \(b = A_0/\lambda L_0\). We attempt to solve the case when the upper bound of the backorder ratio \(\beta_0 = 0.95\), and the scaling parameter \(\lambda = 0.75, 1.00, 1.25, 2.50, 5.00\). Applying the Algorithm procedure yields the results as tabulated in Table 2. From this table, the optimal inventory policy can easily be found by comparing \(EAC^W(Q_i, k_i, \pi_{x_i}, L_i)\), for \(i = 0,1,2,3\), and thus we summarize these in Table 3. Furthermore, in order to observe the effect of lead time reduction with interaction of ordering cost, we list the result of
fixed ordering cost model (i.e., take $\lambda = \infty$) in the same table. From the results shown in Table 3, we see that as the value of $\lambda$ decreases, the larger savings of total expected annual cost are obtained (comparing the result with fixed ordering cost model). And it is interesting to observe that decreasing the value $\lambda$ will result in a decrease in the total expected annual cost, the order quantity and the backorder price discount, but the reorder point is increasing because the smaller value of $\lambda$ accompanies the smaller ordering cost and the higher safety factor.

**Example 2.** The data is the same as in Example 1, and we assume that the lead time and ordering cost reductions act dependently with the following relationship Chen et al. (2001): $A_0 - A_n/A_0 = \mu \ln(L/L_0)$, which implies $A(L) = f + g \ln L$, where $\mu < 0$, $f = A_0(1 + \mu \ln L_0)$ and $g = -\mu A_0 > 0$. We solve the case when the upper bounds of the backorder ratio $b_0 = 0.95$, $m = C_0 0.2$, $C_0 0.5$, $C_0 0.8$, $C_0 1$. Utilizing the similar procedure as in the Algorithm, we obtain the results in Table 4. From this table, the optimal

<table>
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<tr>
<th>Table 1</th>
<th>Lead time data</th>
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<tr>
<td>Lead time component $i$</td>
<td>Normal duration $b_i$ (days)</td>
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<tr>
<td>1</td>
<td>20</td>
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<tr>
<td>2</td>
<td>20</td>
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<table>
<thead>
<tr>
<th>Table 2</th>
<th>Solution procedures of Example 1 ($L_i$ in weeks)</th>
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<tbody>
<tr>
<td>$\lambda$</td>
<td>$i$</td>
</tr>
<tr>
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<table>
<thead>
<tr>
<th>Table 3</th>
<th>Summary of the optimal solution of Example 1 ($L_i*$ in weeks)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$L_i*$</td>
</tr>
<tr>
<td>0.75</td>
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</tr>
<tr>
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<td>3</td>
</tr>
<tr>
<td>1.25</td>
<td>3</td>
</tr>
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<td>3</td>
</tr>
<tr>
<td>5.00</td>
<td>3</td>
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</table>

Note: saving is based on the fixed ordering cost model (i.e., $\lambda = \infty$).
inventory policy can easily be found by comparing EAC$^w(Q_i, k_i, \pi_{x_i}, L_i)$, for $i = 0, 1, 2, 3$, and thus we summarize these in Table 5. Furthermore, in order to see that the interaction of reduction between the lead time and the ordering cost, we list the result of fixed ordering cost model (i.e., take $m = 0$) in the same table. From the results shown in Table 5, we see that as the value of $m$ decreases, the larger savings of total expected annual cost are obtained (comparing the result with fixed ordering cost model). And it is interesting to observe that decreasing the value $m$ will results in a decrease in the total expected annual cost, the order quantity and the backorder price discount, but the reorder point is increasing because the smaller value of $m$ accompanies the smaller ordering cost and the higher safety factor.

If we know the lead time demand $X$ follows a particular distribution (e.g., normal distribution) which has a p.d.f. $f_X(x)$ with finite mean $DL$ and standard deviation $\sigma \sqrt{L}$, we may recall that $r = DL + k_0 \sigma \sqrt{L}$, and the expected demand shortage at the end of the cycle $E(X - r)^+ = \int_{-\infty}^{\infty} (x - r)f_X(x)dx = \sigma \sqrt{L}\psi(k)$, where $\psi(k) = \phi(k) - k[1 - \Phi(k)] > 0$, $\phi(k)$ and $\Phi(k)$ denote the standard normal probability density function (p.d.f.) and distribution function (d.f.), respectively. Hence, model (2) can be transformed into the following formulation:

$$EAC^N(Q, k, \pi_x, L) = \frac{A(L)}{Q} + h\left[\frac{Q}{2} + k\sigma \sqrt{L}\right]$$

Table 4
Solution procedures of Example 2 ($L_i$ in weeks)

<table>
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<tr>
<th>$\mu$</th>
<th>$i$</th>
<th>$L_i$</th>
<th>$C(L_i)$</th>
<th>$A_i$</th>
<th>$Q_i$</th>
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<th>$k_i$</th>
<th>EAC$^w(Q_i, k_i, \pi_{x_i}, L_i)$</th>
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Table 5
Summary of the optimal solution of Example 2 ($L^*$ in weeks)

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<tr>
<th>$\mu$</th>
<th>$L^*$</th>
<th>$A^*$</th>
<th>$Q^*$</th>
<th>$\pi_{x^*}$</th>
<th>$k^*$</th>
<th>$r^*$</th>
<th>EAC$^w(Q^<em>, k^</em>, \pi_{x^<em>}, L^</em>)$</th>
<th>Saving (%)</th>
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<td>77.60</td>
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<td>63.79</td>
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<td>2.73</td>
<td>67.78</td>
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<td>24.46</td>
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</table>

Note: saving is based on the fixed ordering cost model (i.e., $\mu = 0$).
where $EAC^N(Q, k, \pi_x, L)$ is the total expected annual cost for the normal distribution case and $G(\pi_x) = \pi_0 - \beta_0 \pi_x + \beta_0 \pi_x^2/\pi_0$.

Once again, the approach employed in the previous section is utilized to solve Eq. (15), and then we can get an optimal total expected annual cost $EAC^N(\hat{Q}, \hat{k}, \hat{\pi}_x, \hat{L})$. Hence, as the particular distribution is a normal distribution, the added cost by the standard procedure is $EAC^N(Q^*, k^*, \pi_x^*, L^*) - EAC^N(\hat{Q}, \hat{k}, \hat{\pi}_x, \hat{L})$. This is the largest amount that we would be willing to pay for the knowledge of p.d.f. $f_X$. This quantity can be regarded as the expected value of additional information (EVAI). By Example 3, we can find that the proposed minimax distribution free is a valuable method.

**Example 3.** The data is the same as in Example 1. For the case of $\beta_0 = 0.95$, and the scaling parameter $\lambda = 5.00$, we compare the procedure for the worst case distribution against the normal distribution. In this case, $(Q^*, k^*, \pi_x^*, L^*) = (150.89, 2.25, 77.51, 3)$, $(\hat{Q}, \hat{k}, \hat{\pi}_x, \hat{L}) = (122.90, 1.80, 77.04, 3)$ and the total expected annual cost $EAC^N(150.89, 2.25, 77.51, 3) = 3,002.49$, $EAC^N(122.90, 1.80, 77.04, 3) = 2,898.47$ then $EAC^N(150.89, 2.25, 77.51, 3) - EAC^N(122.90, 1.80, 77.04, 3) = 104.02$.

**5. Concluding remarks**

The purpose of this paper is to extend the Pan and Hsiao’s (2001) model by simultaneously optimizing the order quantity, reorder point, lead time and backorder price, and the reduction of lead time accompanies a decrease of ordering cost. In this paper, we only assume that the first and second moments of the lead time demand are known and finite, but its probability distribution is unknown. And apply the minimax distribution free procedure to find the optimal solution. The results of the numerical examples indicating that when the reduction of lead time accompanies a decrease of ordering cost and larger savings of total expected annual cost can be realized.

In future research on this problem, it would be interesting to deal with an arrival order lot including some defective items. Another possible extension of this work may be conducted by considering that the functional relationships of lead time and ordering cost reductions are other functional forms.

**Acknowledgments**

The author greatly appreciates the anonymous referees for several helpful comments and suggestions on an earlier version of the paper.

**Appendix**

For a given value of $L$, we first obtain the Hessian matrix $H$ as follows:

$$H = \begin{bmatrix}
\frac{\partial^2 EAC^W(Q, k, \pi_x, L)}{\partial Q^2} & \frac{\partial^2 EAC^W(Q, k, \pi_x, L)}{\partial Q \partial \pi_x} & \frac{\partial^2 EAC^W(Q, k, \pi_x, L)}{\partial Q \partial k} \\
\frac{\partial^2 EAC^W(Q, k, \pi_x, L)}{\partial \pi_x \partial Q} & \frac{\partial^2 EAC^W(Q, k, \pi_x, L)}{\partial \pi_x^2} & \frac{\partial^2 EAC^W(Q, k, \pi_x, L)}{\partial \pi_x \partial k} \\
\frac{\partial^2 EAC^W(Q, k, \pi_x, L)}{\partial k \partial Q} & \frac{\partial^2 EAC^W(Q, k, \pi_x, L)}{\partial k \partial \pi_x} & \frac{\partial^2 EAC^W(Q, k, \pi_x, L)}{\partial k^2}
\end{bmatrix},$$

where

$$\frac{\partial^2 EAC^W(Q, k, \pi_x, L)}{\partial Q^2} = \frac{2A(L)D}{Q^4} + \frac{DG(\pi_x)}{Q^3} + \frac{2D}{Q^3}C(L),$$
\[
\frac{\partial^2 \text{EAC}^W(Q, k, \pi_x, L)}{\partial \pi_x^2} = \frac{D\beta_0}{Q\pi_0} \sigma \sqrt{L(\sqrt{1 + k^2} - k)},
\]

\[
\frac{\partial^2 \text{EAC}^W(Q, k, \pi_x, L)}{\partial k^2} = \frac{1}{2} \left[ h \left( 1 - \frac{\beta_0 \pi_x}{\pi_0} \right) + \frac{D}{Q} G(\pi_x) \right] \sigma \sqrt{L(1 + k^2)^{-3/2}},
\]

\[
\frac{\partial^2 \text{EAC}^W(Q, k, \pi_x, L)}{\partial Q \partial \pi_x} = \frac{\partial^2 \text{EAC}^W(Q, k, \pi_x, L)}{\partial \pi_x \partial Q} = \frac{-D}{2Q^2} \left( \frac{2\beta_0 \pi_x}{\pi_0} - \beta_0 \right) \sigma \sqrt{L(\sqrt{1 + k^2} - k)},
\]

\[
\frac{\partial^2 \text{EAC}^W(Q, k, \pi_x, L)}{\partial k \partial \pi_x} = \frac{\partial^2 \text{EAC}^W(Q, k, \pi_x, L)}{\partial \pi_x \partial k} = \frac{1}{2} \left[ h \frac{\beta_0}{\pi_0} - \frac{D}{Q} \left( \frac{2\beta_0 \pi_x}{\pi_0} - \beta_0 \right) \right] \sigma \sqrt{L \left( 1 - \frac{k}{\sqrt{1 + k^2}} \right)},
\]

and

\[
\frac{\partial^2 \text{EAC}^W(Q, k, \pi_x, L)}{\partial Q \partial k} = \frac{\partial^2 \text{EAC}^W(Q, k, \pi_x, L)}{\partial k \partial Q} = \frac{DG(\pi_x)}{2Q^2} \sigma \sqrt{L \left( 1 - \frac{k}{\sqrt{1 + k^2}} \right)}.\]

Next, we proceed by evaluating the principal minor determinant of H at point \((Q^*, k^*, \pi_x^*)\). The first principal minor determinant of H is

\[
|H_{11}| = \frac{2A(L)D}{Q'^3} + \frac{DG(\pi_x^*)}{Q'^3} \sigma \sqrt{L(\sqrt{1 + k'^2} - k^*)} + \frac{2D}{Q'^3} C(L) > 0.
\]

The second principle minor determinant of H is

\[
|H_{22}| = \left[ \frac{2A(L)D}{Q'^3} + \frac{DG(\pi_x^*)}{Q'^3} \sigma \sqrt{L(\sqrt{1 + k'^2} - k^*)} + \frac{2D}{Q'^3} C(L) \right] \left[ \frac{D\beta_0}{Q'\pi_0} \sigma \sqrt{L(\sqrt{1 + k'^2} - k^*)} \right] - \left\{ \frac{D^2}{4Q'^3} \left( \frac{2\beta_0 \pi_x^*}{\pi_0} - \beta_0 \right)^2 \left[ \sigma \sqrt{L(\sqrt{1 + k'^2} - k^*)} \right]^2 \right\}
\]

\[
= \frac{2D^2\beta_0}{Q'^3} \sigma \sqrt{L(\sqrt{1 + k'^2} - k^*)}[A(L) + C(L)]
\]

\[
+ \frac{D^2\beta_0}{Q'^3} \left[ \sigma \sqrt{L(\sqrt{1 + k'^2} - k^*)} \right]^2 \left( 1 - \frac{\beta_0}{4} \right) > 0.
\]

The third principle minor determinant of H is (noting that \(\pi_x^* = (hQ^*/2D) + (\pi_0/2)\))

\[
|H_{33}| = \left\{ \frac{2D^2\beta_0}{Q'^3\pi_0} \sigma \sqrt{L(\sqrt{1 + k'^2} - k^*)}[A(L) + C(L)] + \frac{D^2\beta_0}{Q'^4} \left[ \sigma \sqrt{L(\sqrt{1 + k'^2} - k^*)} \right]^2 \left( 1 - \frac{\beta_0}{4} \right) \right\}
\]

\[
\times \frac{1}{2} \left[ h \left( 1 - \frac{\beta_0 \pi_x^*}{\pi_0} \right) + \frac{D}{Q} G(\pi_x^*) \right] \sigma \sqrt{L(1 + k^2)^{-3/2}} - \left[ \frac{DG(\pi_x^*)}{2Q'^3} \sigma \sqrt{L \left( 1 - \frac{k^*}{\sqrt{1 + k^2}} \right)} \right]^2 \frac{D\beta_0}{Q'\pi_0} \sigma \sqrt{L(\sqrt{1 + k'^2} - k^*)}
\]

\[
= \frac{D^2\beta_0}{Q'^3\pi_0} \sigma \sqrt{L(\sqrt{1 + k'^2} - k^*)}[A(L) + C(L)] \left[ h \left( 1 - \frac{\beta_0 \pi_x^*}{\pi_0} \right) + \frac{D}{Q} G(\pi_x^*) \right] \sigma \sqrt{L(1 + k^2)^{-3/2}}
\]

\[
+ \frac{D^2\beta_0}{4Q'^3} \sigma^2 L(\sqrt{1 + k'^2} - k^*)^2 \left( 4 - \beta_0 \right) \frac{1}{2} h \left( 1 - \frac{\beta_0 \pi_x^*}{\pi_0} \right) \sigma \sqrt{L(1 + k^2)^{-3/2}}
\]

\[
+ \frac{D^2\beta_0}{4Q'^3} \sigma^2 L(\sqrt{1 + k'^2} - k^*)^2 \left( 4 - \beta_0 \right) \frac{1}{2} \frac{D}{Q} G(\pi_x^*) \sigma \sqrt{L(1 + k^2)^{-3/2}}.
\]
\[
- \left[ \frac{D^2 \beta_0^2 (\pi \pi^*)}{4Q^4} \alpha^2 L \left( 1 - \frac{k^*}{\sqrt{1+k^*}} \right)^2 \right] \frac{D \beta_0}{Q} \sqrt{L} \sqrt{1+k^*}^2 - k^*
\]
\[
> \frac{D^3 \beta_0}{8Q^6} \alpha^2 L \left( 1 + k^* - k^* \right)^2 \left( 4 - \beta_0 \right) G(\pi^*) \sqrt{L} (1+k^*)^{-3/2}
\]
\[
- \left[ \frac{D^2 G^2 (\pi \pi^*)}{4Q^4} \alpha^2 L \left( 1 - \frac{k^*}{\sqrt{1+k^*}} \right)^2 \right] \frac{\beta_0}{\pi_0} \sqrt{L} \sqrt{1+k^*} - k^*
\]
\[
= \frac{D^3 \alpha^2 L}{8Q^6} \alpha \sqrt{L} \sqrt{1+k^*} - k^* G(\pi^*)
\]
\[
\times \left\{ \beta_0 \left( 1 + k^* - k^* \right)^2 \left( 4 - \beta_0 \right) \sqrt{1+k^*}^2 - k^* \right\}
\]
\[
\times 2 \left( 1 - \frac{k^*}{\sqrt{1+k^*}} \right)^2 \left[ \frac{\beta_0 \pi^*}{\pi_0} (\pi^* - \pi_0) + \beta_0 \right] \}
\]
\[
= \frac{D^3 \alpha^2 L}{8Q^6} \alpha \sqrt{L} \sqrt{1+k^*} - k^* G(\pi^*)
\]
\[
\times \left[ \beta_0 \left( 2 - \beta_0 \right) \left( \frac{1 + k^* - k^*}{(1+k^*)^{3/2}} \right) + 2 \beta_0 \left( \frac{1 + k^* - k^*}{(1+k^*)^{3/2}} \right) - 2 \beta_0 \left( \frac{1 + k^* - k^*}{(1+k^*)^{3/2}} \right) \right] \]
\[
> \frac{D^3 \sigma^2 L}{8Q^6} \alpha \sqrt{L} \sqrt{1+k^*} - k^* G(\pi^*) \left[ 2 \beta_0 \left( \frac{1 + k^* - k^*}{(1+k^*)^{3/2}} \right) - \frac{(1 + k^* - k^*)}{1+k^*} \right]
\]
\[
= \frac{D^3 \sigma^2 L}{8Q^6} \alpha \sqrt{L} \sqrt{1+k^*} - k^* G(\pi^*) \left[ 2 \beta_0 \frac{\sqrt{1 + k^*} - k^*}{(1+k^*)^{3/2}} \left( k^* \sqrt{1+k^*} - k^* \right) \right] > 0.
\]

Hence, \( |H_{33}| > 0 \).

Therefore, the Hessian matrix \( H \) is positive definite at point \((Q^*, k^*, \pi^*)\). The proof is completed.

References


Chiu, P.P., 1998. Economic production quantity models inventory involving lead time as a decision variable. Master Thesis, National Taiwan University of Science and Technology.
